PROBLEM CONCERNING A SPHERICAL PISTON IN A COMPRESSIBLE MEDIUM WITH "DRY" FRICTION

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We study the self-similar problem concerning the motion of a spherical piston in a medium with "dry" friction. The piston moves with constant velocity in a nonideal medium.

1. We consider a spherical piston which begins its motion from the coordinate origin, moving with constant speed in a medium with the equation of state

$$
\begin{gathered}
-{}^1/_3\, 3(\sigma^r + 2\sigma^\theta) = K(\rho/\rho_0 - 1)^\gamma, \quad \gamma \geqslant 1, \quad \rho \geqslant \rho_0 \\
\sigma^r + 2\sigma^\theta = 0, \quad \rho < \rho_0 \\
{}^1/_2(\sigma^r - \sigma^\theta) = \varkappa p, \qquad \varkappa < 0
\end{gathered} \tag{1.1}
$$

Here σ^r and σ^θ are, respectively, the radial and azimuthal stresses, p is the pressure, ρ and ρ_0 are densities, K is the volume compression coefficient, $\varkappa$ is the coefficient of dry friction, and u_p is the speed of the piston.

Similar equations of state were considered in [1, 2].

The equations of motion have the form

$$
\begin{gathered}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = \frac{1}{\rho}\frac{\partial \sigma^r}{\partial r} + \frac{2(\sigma^r - \sigma^\theta)}{\rho r} \\
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2u\rho}{r} = 0
\end{gathered} \tag{1.2}
$$

The initial and boundary conditions for Eqs. (1.1) and (1.2) have the form

$$
\begin{gathered}
u = p = 0, \quad \rho = \rho_0 \ \text{for} \ t = 0, \quad r \geqslant 0 \\
u = u_p \ \text{for} \ r = u_p t \\
u \to 0, \quad p \to 0, \quad \rho \to \rho_0 \ \text{for} \ r \to \infty
\end{gathered}
$$

Dimensional analysis [3] shows that the problem is self-similar. We introduce the dimensionless quantities

$$
\begin{gathered}
\lambda = rt^{-1}\left[\frac{\rho_0}{(1 - {}^4/_3\varkappa)K}\right]^{1/2}, \qquad U = u\left(\frac{\rho_0}{(1 - {}^4/_3\varkappa)K}\right)^{1/2} \\
P = p \cdot K^{-1}, \quad R = \rho\rho_0^{-1}, \quad \alpha = 4\varkappa(1 - {}^4/_3\varkappa)^{-1}
\end{gathered}
$$

In these variables the system (1.2) assumes the form

$$
\begin{gathered}
\frac{dU}{d\lambda}\left(\frac{U}{\lambda} - 1\right) = -\frac{1}{\lambda R}\frac{dP}{d\lambda} + \frac{\alpha}{\lambda^2}\frac{P}{R} \\
\frac{dR}{d\lambda}\left(\frac{U}{\lambda} - 1\right) + \frac{R}{\lambda}\frac{dU}{d\lambda} + 2\frac{RU}{\lambda^2} = 0 \\
P = (R - 1)^\gamma
\end{gathered} \tag{1.3}
$$

We solve the system (1.3) for the derivatives wherein we use the third of the relations (1.3)

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$$\frac{dU}{d\lambda} = \lambda^{-1}\frac{\gamma^{-1}\alpha(U-\lambda)P^{1/\gamma}(P^{1/\gamma}+1)^{-1}+2U}{\gamma^{-1}(U-\lambda)^2P^{1/\gamma-1}-1} \tag{1.4}$$

$$\frac{dP}{d\lambda} = -\lambda^{-1}\frac{2U(U-\lambda)(P^{1/\gamma}+1)+\alpha P}{\gamma^{-1}(U-\lambda)^2P^{1/\gamma-1}-1}$$

The boundary conditions for equations (1.4) have the form

$$U=\lambda \quad \text{for} \quad \lambda = u_p\left(\frac{\rho_0}{(1-{}^4/_3\varkappa)K}\right)^{1/2}$$

$$U\to 0,\ P\to 0 \quad \text{for} \quad \lambda\to\infty.$$

2. Taking equations (1.1) into account, we can write the conditions at a strong discontinuity [4] in the form

$$(P_1^{1/\gamma}+1)(U_1-\lambda) = (P_2^{1/\gamma}+1)(U_2-\lambda) \tag{2.1}$$

$$P_1-(P_1^{1/\gamma}+1)U_1(\lambda-U_1) = P_2-(P_2^{1/\gamma}+1)U_2(\lambda-U_2)$$

If the front of the shock wave propagates through a quiescent medium, we have $P_2 = 0$, $U_2 = 0$ and the conditions at the discontinuity assume the form

$$(P_1^{1/\gamma}+1)(U_1-\lambda) = -\lambda,\quad P_1-(P_1^{1/\gamma}+1)U_1(\lambda-U_1) = 0$$

A degenerate strong discontinuity ($P_1\to 0$, $U_1\to 0$) yields a value of λ_c corresponding to the propagation speed of a weak discontinuity. Some calculations show that

$$\gamma>1,\ \lambda_c=0;\qquad \gamma=1,\ \lambda_c=1$$

Eliminating P_1 from equations (2.1), we obtain a curve in the λU plane. It is not difficult to show that this curve lies below the "initial data" line, i.e., the line $\lambda = U$. Indeed, from the first of the relations (2.1) it is evident that for $\lambda > 0$, $P_1 > 0$, we have $U_1 < \lambda$.

The singular points of the system of equations (1.4) are determined by the system of algebraic equations

$$\lambda[(U-\lambda)^2P^{1/\gamma-1}\gamma^{-1}-1]=0$$

$$\gamma^{-1}\alpha(U-\lambda)P^{1/\gamma}(P^{1/\gamma}+1)^{-1}+2U=0 \tag{2.2}$$

$$2U(U-\lambda)(P^{1/\gamma}+1)+\alpha P=0$$

Let $\lambda = 0$. Then

$$\gamma^{-1}\alpha UP^{1/\gamma}(P^{1/\gamma}+1)^{-1}+2U=0,\qquad 2U^2(P^{1/\gamma}+1)+\alpha P=0$$

If $U = 0$, then $P = 0$. If $U \neq 0$, we have

$$\gamma^{-1}\alpha P^{1/\gamma}=-2(P^{1/\gamma}+1),\qquad P^{1/\gamma}=-2(\gamma^{-1}\alpha+2)^{-1} \tag{2.3}$$

$$U=\pm\gamma^{-1/2} \tag{2.4}$$

For some values of α and γ, Eqs. (2.3) and (2.4) define real singular points.

Assume that $\lambda \neq 0$. In this case, multiplying the second of equations (2.2) by $(U-\lambda)$ and using the first equation, we find that the third equation is a consequence of the first two. Thus, in this case the singular points are determined by the system of equations

$$\gamma^{-1}(U-\lambda)^2P^{1/\gamma-1}-1=0,\quad 2U(U-\lambda)(P^{1/\gamma}+1)+\alpha P=0 \tag{2.5}$$

and form a singular curve.

The singular points $(0,0,0)$, $(0, \pm\gamma^{-1/2}[-2(\alpha\gamma^{-1}+2)^{-1}]^{1/2(\gamma-1)}, [-2(\alpha\gamma^{-1}+2)^{-1}]^{\gamma})$ lie outside the region of flow and have an influence only on the location of the integral curves in the λUP space; no single trajectory (except for the trajectory corresponding to the equilibrium state) passes through these points.

Consider now the behavior of the "singular" curve. From equations (2.5) it follows that the relation

$$\gamma^{\gamma/(1-\gamma)}(U-\lambda)^{2\gamma/(\gamma-1)}=P \tag{2.6}$$

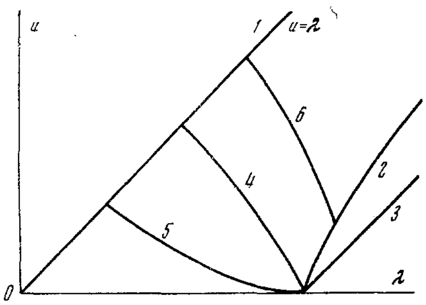

Fig. 1

is satisfied on the singular curve.

From equations (2.1) we find, after some simple manipulations, that on the shock wave

$$(\lambda - U)^2 = P^{(\gamma-1)/\gamma}(P^{1/\gamma} + 1)^{-1}$$

A trajectory starting at the line $U - \lambda = 0$ does not intersect the singular curve before it intersects the strong discontinuity curve. Indeed, let P_0 be the initial pressure. By virtue of the equations (1.4),

$$\frac{dP}{d\lambda} < 0, \qquad \frac{d(\lambda - U)}{d\lambda} > 0$$

Let us calculate the value of the right side of Eq. (2.6) for

$$(\lambda - U)^2 = P^{(\gamma-1)/\gamma}(P^{1/\gamma} + 1)^{-1}$$

This value is equal to

$$\gamma^{\gamma/(1-\gamma)} P (P^{1/\gamma} + 1)^{2\gamma/(1-\gamma)}$$

Since $\gamma > 1$, this value is less than P, i.e., Eq. (2.6) cannot be satisfied at any point of a trajectory preceding the shock wave.

In Fig. 1 we display the "initial data" line 1, the shock wave 2, the singular curve 3, and several characteristic trajectories with $\gamma = 1$: the separatrix 4, a continuous solution 5, and the solution with the shock wave 6. It is clear that a continuous flow region is impossible with $\gamma > 1$. However, with $\gamma = 1$ there is a possibility of such a flow, trajectories of which were uncovered by numerically solving Eq. (1.4). We discuss the numerical results in Sec. 4. From now on we consider the case $\gamma = 1$.

3. For $\gamma = 1$, equations (1.4) assume the form

$$dU/d\lambda = [\alpha(U - \lambda)P(P + 1)^{-1} + 2U]\lambda[(U - \lambda)^2 - 1]$$
$$dP/d\lambda = -[2U(U - \lambda)(P + 1) + \alpha P]\lambda[(U - \lambda)^2 - 1] \tag{3.1}$$

The singular curve is determined as the solution of the system

$$(U - \lambda)^2 = 1, \quad 2U(U - \lambda)(P + 1) + \alpha P = 0$$

In the λU plane the singular curve breaks up into two lines of which the line $U = \lambda - 1$ is of interest. On this line only the state

$$\lambda = 1, \quad U = 0, \quad P = 0$$

is physically realizable. At other points $P < 0$. Thus the point (1, 0, 0) is a singular point through which a trajectory can pass. We note that this point corresponds to a weak discontinuity; at this point we can "splice together" a quiescent region and a perturbed region.

We examine the behavior of the integral curves in a neighborhood of the singular point. In accord with [5], instead of the system (3.1), we consider the system

$$dU/d\tau = \alpha(U - \lambda)P(P + 1)^{-1} + 2U$$
$$dP/d\tau = -2U(U - \lambda)(P + 1) - \alpha P \tag{3.2}$$
$$d\lambda/d\tau = \lambda[(U - \lambda)^2 - 1]$$

and we expand the solution in a series in a neighborhood of the point (1, 0, 0). The linearized system has the form

$$\frac{dU}{d\tau} = -\alpha P + 2U, \qquad \frac{dP}{d\tau} = 2U - \alpha P, \qquad \frac{d\delta}{d\tau} = -2U + 2\delta \tag{3.3}$$

Here $\delta = \lambda - 1$. The characteristic numbers of the system (3.3), written in increasing order, are the three numbers $-0, 2, 2 - \alpha$. To each characteristic number there corresponds a solution of the system (3.3). These solutions are easy to obtain; in vector form they are

$$Y_1 = (1, 2/\alpha, 1), \qquad Y_2 = (0, 0, e^{2\tau}), \qquad Y_3 = (e^{(2-\alpha)\tau}, e^{(2-\alpha)\tau}, 2\varkappa^{-1}e^{(2-\alpha)\tau})$$

The general solution of the system (3.3) may be represented in the form

$$Y = C_1Y_1 + C_2Y_2 + C_3Y_3$$

Integral curves come into the point $\lambda = 1$, $U = 0$, $P = 0$ only for $C_1 = 0$. Thus, in a neighborhood of the singular point, integral curves which can correspond to a motion have the form

$$U = C_3e^{(2-\alpha)\tau}, \qquad P = C_3e^{(2-\alpha)\tau}, \qquad \delta = C_2e^{2\tau} + C_3 2\varkappa^{-1}e^{(2-\alpha)\tau}$$

Trajectories enter the singular point for $\tau \to \infty$. It is evident that $U = P$ in the neighborhood of the singular point, i.e., the line $U = P$ is a separatrix in the UP plane. We find the dependence of U on λ. We have $(2 - \alpha)\tau = \ln(U/C_3)$, then

$$\lambda - 1 = C_2(U/C_3)^{2/(2-\alpha)} + 2\alpha^{-1}U,$$
$$d\lambda / dU = C_2(U/C_3)^{\alpha/(2-\alpha)}C_3^{-1} + 2\alpha^{-1}$$

Since $\alpha < 0$, then for $C_2 \neq 0$, $U \to 0$, $d\lambda/dU \to \infty$, and all the integral curves are tangent to the line $U = 0$. If $C_2 = 0$, then $\lambda = 1 + 2\alpha^{-1}U$, and this equation determines the direction (second separatrix) along which a single trajectory comes into the point (1, 0, 0).

Thus, if a continuous flow regime is realized, the flow almost always takes place without a weak discontinuity.

Actually, since almost all (except for one) of the trajectories come into the point (1, 0, 0) with zero slope ($dU/d\lambda = 0$), they join with the zero solution with no discontinuity in the first derivative of the speed; there is no discontinuity in $dP/d\lambda$ since

$$\lim_{\lambda \to 1-0} \frac{dP}{d\lambda} = \lim \frac{dP}{dU}\frac{dU}{d\lambda} = 0$$

4. In the numerical calculations we first constructed a solution, for various values of α, which comes into the point (1, 0, 0) with a zero slope. The point of intersection of the integral curve with the initial data line determines the value of the piston speed u_p* separating two essentially distinct flow regimes: one with a shock wave and one without. It is clear that as α increases (i.e., as the coefficient $\varkappa$ of dry friction increases) the value of u_p* increases and approaches a limiting value.

In the second stage of our calculations we determined, for various values of α, how the pressure on the piston varied with its speed. For a value of $\alpha = -1.75$ we found the following dependence:

U_p	0.25	0.53	0.56	0.75	1.06
P_p	0.23	0.58	0.62	0.92	1.20

In the third stage, for $\alpha = -1.75$ we determined how the speed of the perturbation front varied with the piston speed:

U_p	0.25	0.53	0.56	0.79	1.06	1.18
U_f	1	1	1	1.1	1.3	1.4

Thus we established that for a given free-flowing medium there is, for small piston speeds, a continuous flow without weak discontinuities; at some speed a weak discontinuity occurs; as the speed increases further a shock wave results.

We note, in conclusion, that a similar problem was considered in [2] wherein the medium was assumed to be incompressible; an analogous problem in the planar case was solved in [6].

Difficulties in studying the system of equations (3.1) did not permit an analytical study of the stability and uniqueness of the solutions of this system to be made; however, the results of a numerical study, carried out in connection with a program presented in a report to the First All-Union Seminar on the Theory of Models of Continuum Mechanics [7], confirmed, with good accuracy, the results obtained in an approximate solution of the system (3.1).

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